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Remarks on dissipative processes in the continuum theory of micromagnetics

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Abstract. Dissipative processes resulting from magnetic spin–crystal lattice and spin–spin interactions are examined from the point of view of modern continuum mechanics in the theory of micromagnetics of deformable media. A formula which generalizes that of Gilbert and Kelley is derived for deformable solids. The corresponding heat conduction equation is obtained.

1. Introduction

It is well known that, in the macroscopic theory of magnetically saturated media referred to as *micromagnetics*, the magnetic spin $\boldsymbol{\mu}$ (or magnetization per unit mass) has a motion described by the equations

$$\frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}t} \equiv \dot{\boldsymbol{\mu}} = \gamma_0 \boldsymbol{B}^{\mathrm{eff}} \times \boldsymbol{\mu}, \qquad |\boldsymbol{\mu}(x, y, z, t)| = \text{constant} \tag{1.1}$$

where γ_0 is the gyromagnetic ratio of an electron ($\gamma_0 = -|e|/mc$). The maxwellian magnetic field, the anisotropy field and the exchange forces which result from spin–spin interactions contribute to the effective or equilibrium field $\boldsymbol{B}^{\mathrm{eff}}$. Furthermore, in the case of a moving deformable material, the operator $\mathrm{d}/\mathrm{d}t$ becomes the total time derivative of continuum mechanics and stress and deformation fields must be taken into account in the analysis. However, apart from the slight modification just mentioned, the form of equation (1.1) is not altered. In résumé, if a magnetic moment $\boldsymbol{\mu}$ is considered to be attached to each material point in the solid specimen then, upon application of a magnetic field or tensions on the boundary of the specimen, the material point is displaced (deformation) while the magnetic moment rotates about this point; this follows from equation (1.1). The two phenomena are linked via the moment of momentum equation since the stress tensor is not symmetric. Such a phenomenological theory has been recently proposed by the author in collaboration with A C Eringen (Maugin and Eringen 1972a, b). A somewhat more concise outline including the relativistically invariant theory developed by Maugin and Eringen (1972c) will appear in book form (Maugin 1972a); some results have been enunciated in a brief note (Maugin 1971b). Other works developed in the same spirit are those of Tiersten (1964, 1965), Brown (1966) and Akhiezer *et al* (1967). Let us assume that the changes are slow or the frequencies low in many cases, thus peculiarly dynamical effects may be ignored; the theory was so achieved in the frame of

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quasimagnetostatics with the currents neglected. Dissipative phenomena (although they have been investigated and their existence confirmed—for instance, in magnetic resonance, coherent rotation phenomena and domain wall propagation), were considered as exceptions as well as heat propagation and temperature fluctuations. Thus isothermal conditions were most often assumed since departure from these conditions would result in irreversibility and damping of the rotational motion of the magnetization (cf Brown 1963, p 6). These dissipative phenomena are however important for practical applications and deserve our attention (cf § 2). It is therefore the aim of the present work to complement, still in the spirit of modern continuum mechanics, our preceding articles (Maugin and Eringen 1972a, b) especially as far as dissipation is concerned.

A remark important for the following is in order. In our paper (Maugin and Eringen 1972a, note 48) that treated nonlinear constitutive equations, we wrote 'it is of course possible to introduce very special forms of dissipation in a variational formulation for instance, through a Rayleigh function of dissipation.... However, to the knowledge of the authors, there exists no means to introduce dissipation in terms as general as the potential giving the recoverable parts of the nonlinear constitutive equations'. The last statement is apparently false since, of course, one can use Onsager's relations in order to determine generalized forces linear in the fluxes, this within the framework of a linear constitutive theory. Furthermore, one can make use of the principle of least irreversible force established by Ziegler (1963) for the nonlinear constitutive theory. In some cases such as plasticity and viscoplasticity theories, the latter principle proved to be very useful. Although we shall base the forthcoming development on this principle, we ought to note that it is not generally accepted and some authors have serious objections on its general validity.

In § 2, we recall the physical background and review special types of dissipation like those considered by Gilbert and Kelley and Landau and Lifshitz. Notations and field equations previously derived are recalled in § 3. Generalities such as the second principle of thermodynamics and the principle of least irreversible force of Ziegler are dealt with in § 4.1. Special types of dissipation which encompass forms previously known are studied in the two last paragraphs 4.2 and 4.3.

2. Physical background

The equation (1.1) describes a rotation of $\boldsymbol{\mu}$ with an angular velocity $\boldsymbol{\omega} = \gamma_0 \boldsymbol{B}^{\text{eff}}$ in the plane formed by the magnetization $\boldsymbol{\mu}$ at the initial time $t = t_0$ and the effective field $\boldsymbol{B}^{\text{eff}}$ of the equilibrium configuration. The latter field we can write as (cf Maugin and Eringen 1972a)

$$\boldsymbol{B}^{\text{eff}} = \boldsymbol{B} + \mathscr{B} + \rho^{-1} \nabla \cdot \boldsymbol{\tau} \tag{2.1}$$

where ρ is the matter density in the deformed configuration of the material, $\boldsymbol{B}$ is the maxwellian magnetic field (the solution to Maxwell's equations for quasimagnetostatics), $\mathscr{B}$ is the anisotropy field and $\boldsymbol{\tau}$ is the spin–spin interaction tensor, a second order tensor which takes account phenomenologically of the interactions between neighbouring spins, an effect of quantum mechanical origin. In fact the rotation of the magnetization observed upon a perturbation or the switching on of an applied magnetic field does not take place in a plane. The motion of a magnetic spin is very much complicated by the influence of interactions with its surroundings. The magnetization spirals into parallelism with the effective field (cf Anderson 1968, p 180). It relaxes to its equilibrium position. This fact is accounted for as follows. The spiralling of the magnetization is a dissipative phenomenon similar to internal friction. The temperature fluctuations which are of stochastic nature give rise phenomenologically to a random field $\boldsymbol{B}^{\text{rand}}$ whose statistical average is zero (cf Brown 1959). The equation (1.1) should therefore be modified to read

$$\dot{\boldsymbol{\mu}} = \gamma_0(\boldsymbol{B}^{\text{eff}} + \boldsymbol{B}^{\text{rand}}) \times \boldsymbol{\mu}. \tag{2.2}$$

This is the Langevin equation of the magnetization motion. But, as is well known in the theory of brownian motion, the random force acting on a brownian particle is necessarily combined with a frictional force, a fact which represents a very general law of nature (in this regard, see the fluctuation–dissipation theorem in Kubo 1966). A possible friction term has been proposed by Gilbert and Kelley (1955). Adding this term to equation (2.2), with α a damping coefficient and a factor μ_S^{-1} introduced for dimensional convenience, the spin equation is written

$$\dot{\boldsymbol{\mu}} = \gamma_0(\boldsymbol{B}^{\text{eff}} + \boldsymbol{B}^{\text{rand}}) \times \boldsymbol{\mu} - \alpha\gamma_0\mu_S^{-1}\dot{\boldsymbol{\mu}} \times \boldsymbol{\mu}. \tag{2.3}$$

The frictional force is related in some way to the random field $\boldsymbol{B}^{\text{rand}}$ by the fluctuation–dissipation theorem. However, compared to the familiar theory of brownian motion of a free particle or a harmonic oscillator, the brownian motion of spin involves some complexities. The latter come from the quasi-nonlinear structure of equation (2.3). As emphasized by Kubo and Hashitsume (1970), a simple harmonic analysis cannot be employed since the magnetization is in general not linear in the stochastic field $\boldsymbol{B}^{\text{rand}}$. Even if the basic process $\boldsymbol{B}^{\text{rand}}$ is assumed to be gaussian, that is, relatively simple, equation (2.3) is hard to solve.

If we discard the stochastic field $\boldsymbol{B}^{\text{rand}}$, equation (2.3) still describes the approach of $\boldsymbol{\mu}$ to $\boldsymbol{B}^{\text{eff}}$ in a sufficient realistic manner and is known to give a good description of loss mechanisms in a number of applications in ferromagnetic resonance (cf Anderson 1968, p 180). Thus

$$\dot{\boldsymbol{\mu}} = \gamma_0 \boldsymbol{B}^{\text{eff}} \times \boldsymbol{\mu} - \alpha\gamma_0\mu_S^{-1}\dot{\boldsymbol{\mu}} \times \boldsymbol{\mu}. \tag{2.4}$$

Another form of dissipation has been proposed earlier by Landau and Lifshitz (1935). For a certain approximation, their equation can be deduced from equation (2.4). Indeed, for small damping constant α, we can replace $\dot{\boldsymbol{\mu}}$ in the last term of equation (2.4) by its value given by equation (1.1) and, setting $\beta = -\alpha\gamma_0^2\mu_S^{-1}$, we obtain

$$\dot{\boldsymbol{\mu}} = \gamma_0 \boldsymbol{B}^{\text{eff}} \times \boldsymbol{\mu} + \beta(\boldsymbol{B}^{\text{eff}} \times \boldsymbol{\mu}) \times \boldsymbol{\mu} \tag{2.5}$$

which is the Landau–Lifshitz equation where $\boldsymbol{B}^{\text{eff}}$ contains in general nonlinear terms. The damping factor involved is determined by the environment of the magnetic spin. If interactions between electron spins and the crystal lattice are the predominant source of damping, the relaxation time τ_1 which, according to the Einstein equation (cf Kubo and Hashitsume 1970), is given by the formula

$$\tau_1 = (2\beta k\theta)^{-1} \tag{2.6}$$

where k is the Boltzmann constant and θ the thermodynamical temperature, is of the order of 10^{-7} s at 78 K and 10^{-2} s at 4·2 K. If spin–spin interactions predominate then τ_1 is of the order of 10^{-9} to 10^{-10} s and can be considered independent of temperature (cf Anderson 1968, p 180). In the former case, τ_1 increases with decreasing temperature.

An equation such as equation (2.4) can be constructed with the help of a variational principle by introducing a special form of Rayleigh dissipation function (cf Maugin and

Eringen 1972a, Maugin 1972a). It is the purpose of this work to show that, by applying general principles of continuum mechanics (second principle of thermodynamics, principle of least irreversible force, invariances), one can arrive at a similar equation. All we need to do is to study the dissipative parts of the constitutive equations. In particular, considering that $\mathscr{B}$ and $\boldsymbol{\tau}$ present additive recoverable and dissipative parts

$$\mathscr{B} = {}^{\mathrm{R}}\mathscr{B} + {}^{\mathrm{D}}\mathscr{B} \qquad \boldsymbol{\tau} = {}^{\mathrm{R}}\boldsymbol{\tau} + {}^{\mathrm{D}}\boldsymbol{\tau} \tag{2.7}$$

equation (1.1) will read

$$\dot{\boldsymbol{\mu}} = \gamma_0(\boldsymbol{B} + {}^{\mathrm{R}}\mathscr{B} + \rho^{-1}\nabla\,.\,{}^{\mathrm{R}}\boldsymbol{\tau}) \times \boldsymbol{\mu} + \gamma_0({}^{\mathrm{D}}\mathscr{B} + \rho^{-1}\nabla\,.\,{}^{\mathrm{D}}\boldsymbol{\tau}) \times \boldsymbol{\mu}. \tag{2.8}$$

Dealing with general dissipative processes, we also have to consider dissipative stresses and caloric dissipation which were excluded from our variational treatment (Maugin and Eringen 1972a).

3. Notation and field equations

3.1. Notation

For the sake of simplicity, we use cartesian coordinates. Capital and lower case italic indices take the values 1, 2 and 3. The summation convention on repeated indices is used throughout. We consider a material body $(B) \subset \mathbb{E}^3$, of boundary (∂B) with unit exterior oriented normal $\boldsymbol{n}$ in the deformed (actual) configuration $\mathscr{K}((B_{\mathrm{R}})$ and $(\partial B_{\mathrm{R}})$ in the undeformed (initial) configuration $\mathscr{K}_{\mathrm{R}})$. $\boldsymbol{x}(x_k)$ and $\boldsymbol{X}(X_K)$ are cartesian eulerian and lagrangian coordinates in $\mathscr{K}$ and $\mathscr{K}_{\mathrm{R}}$ respectively. $\boldsymbol{v}(v_k)$ is the velocity field and $\boldsymbol{u}(u_k)$ is the displacement field. ρ and ρ_{R} are the matter density in $\mathscr{K}$ and $\mathscr{K}_{\mathrm{R}}$ respectively. $\boldsymbol{t}(t_{kl})$ is the nonsymmetric stress tensor (thus the position of indices is important). ϵ, η, ψ and h are the specific internal energy, the specific entropy, the specific free (Helmholtz) energy and the heat supply per unit mass respectively. $\boldsymbol{q}(q_k)$ is the heat flux vector. Commas denote partial differentiation and the total time derivative is denoted by a superscript dot or a symbol $\mathrm{d}/\mathrm{d}t$, for example,

$$x_{k,K} \equiv \frac{\partial x_k}{\partial X_K}, \qquad \frac{\mathrm{d}A}{\mathrm{d}t} \equiv \dot{A} = \frac{\partial A}{\partial t} + v_l A_{,l} \tag{3.1}$$

ϵ_{ijk} is the permutation symbol ($\epsilon_{123} = 1$). Parentheses around a set of indices denote symmetrization while brackets denote alternation, for example

$$d_{kl} = v_{(k,l)} = \tfrac{1}{2}(v_{k,l} + v_{l,k}), \qquad \pi_{kl} = v_{[k,l]} = \tfrac{1}{2}(v_{k,l} - v_{l,k}) \tag{3.2}$$

where d_{kl} is the rate of strain tensor and π_{kl} is the vorticity tensor.

3.2. Field equations

We refer the reader to Maugin and Eringen (1972a) for the derivation of the following local field equations. They have been established in the frame of quasimagnetostatics, the currents being neglected, with no other volume forces than those of magnetic origin. The gyromagnetic effect is taken to be isotropic (magnetic spin proportional to an angular momentum). The set of field equations consists of the following.

(i) Conditions of saturation in (B) and on (∂B)

$$\mu_k \mu_k = \mu_S^2 = \text{constant}$$
$$\mu_k \dot{\mu}_k = 0, \qquad \mu_k \mu_{k,K} = 0. \tag{3.3}$$

(ii) Field equations in the continuous region (B):

(*a*) conservation of mass

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0 \tag{3.4}$$

(*b*) conservation of momentum

$$\rho \dot{v}_k = t_{kl,l} + \rho B_{l,k} \mu_l \tag{3.5}$$

(*c*) conservation of moment of momentum

$$t_{[kl]} = \rho \mathscr{B}_{[k} \mu_{l]} \tag{3.6}$$

(*d*) conservation of magnetic spin momentum

$$\rho \dot{\boldsymbol{\mu}} = \rho \gamma_0 \boldsymbol{B}^{\text{eff}} \times \boldsymbol{\mu} \tag{3.7}$$

(*e*) Maxwell's equations

$$\nabla \times \boldsymbol{B} = \nabla \times \rho \boldsymbol{\mu}, \qquad \nabla . \boldsymbol{B} = 0 \tag{3.8}$$

(*f*) energy equation

$$\rho \dot{\epsilon} = t_{kl} v_{k,l} + \tau_{kl} \dot{\mu}_{k,l} - \rho \mathscr{B}_k \dot{\mu}_k + q_{k,k} + \rho h \tag{3.9}$$

(*g*) entropy inequality

$$\rho \dot{\eta} - (q_k/\theta)_{,k} - (\rho h/\theta) \geqslant 0. \tag{3.10}$$

(iii) Field equations outside (B): that is, in vacuum

$$\nabla \times \boldsymbol{B} = 0, \qquad \nabla . \boldsymbol{B} = 0. \tag{3.11}$$

(iv) Boundary conditions on (∂B)

$$t_{kl} n_l = t_k - \tfrac{1}{2} \rho^{(\text{in})} \boldsymbol{\mu}^{(\text{in})} . (\boldsymbol{B}^{(\text{out})} - \boldsymbol{B}^{(\text{in})}) n_k \tag{3.12}$$

$$\epsilon_{ijk} \tau_{jm}^{(\text{in})} \mu_k^{(\text{in})} n_m = 0 \tag{3.13}$$

$$\boldsymbol{n} \times (\boldsymbol{B}^{(\text{out})} - \boldsymbol{B}^{(\text{in})}) = \rho \boldsymbol{\mu}^{(\text{in})} \times \boldsymbol{n} \tag{3.14}$$

with obvious notations.

On $(\partial B)_t$, a part of (∂B), the purely mechanical traction t_k is given. On $(\partial B)_u$ such that $(\partial B)_u \cup (\partial B)_t = (\partial B)$ and $(\partial B)_u \cap (\partial B)_t = \phi$, the displacement u_k is prescribed. Cauchy's data for the system given above are at $t = t_0$

$$\rho(\boldsymbol{x}, t = t_0) = \rho_R(\boldsymbol{X}), \qquad x_k(t = t_0) = g_{kK}(\boldsymbol{x}, \boldsymbol{X}) X_K$$
$$v_k(\boldsymbol{x}, t = t_0) = v_{(0)k}(\boldsymbol{x}) \tag{3.15}$$
$$\mu_k(\boldsymbol{x}, t = t_0) = g_{kK}(\boldsymbol{x}, \boldsymbol{X}) \mu_K(\boldsymbol{X}), \qquad |\boldsymbol{\mu}| = \mu_S$$

given where $g_{kK}(\boldsymbol{x}, \boldsymbol{X})$ are shifters (cf Eringen 1967) equal to unity at $t = t_0$. μ_K is the initial configuration of $\boldsymbol{\mu}$. The term h is given.

The number of components of the field unknowns ρ, $\boldsymbol{v}$, $\boldsymbol{t}$, $\boldsymbol{\mu}$, $\boldsymbol{B}$, $\boldsymbol{\tau}$, ϵ, θ and $\boldsymbol{q}$ amounts to $1+3+9+3+3+3+9+1+1+3 = 36$, while the number of independent field equations valid in the continuous region (B), equations (3.4) through (3.9), amounts to $1+3+3+2+2+1 = 12$. Therefore we need 24 supplementary equations to render the problem determined. These are provided by the constitutive equations (9 for $\boldsymbol{\tau}$, 3 for $\boldsymbol{\mathscr{B}}$, 9 for $\boldsymbol{t}$ and 3 for $\boldsymbol{q}$).

4. Dissipative processes

4.1. Generalities

In this paragraph, we consider heat conducting nonlinear solids. To start with, the material symmetry is not specified. We examine the thermodynamical restrictions that follow from the local form of the second principle of thermodynamics (3.10). Firstly, however, we shall look at the equation of entropy production. Dividing equation (3.9) by θ and re-arranging some terms, we get

$$\rho\theta^{-1}\dot{\epsilon} = \theta^{-1}(t_{kl}v_{k,l}+\tau_{kl}\dot{\mu}_{k,l}-\rho\mathscr{B}_k\dot{\mu}_k-\theta^{-1}q_k\theta_{,k})+\left(\frac{q_k}{\theta}\right)_{,k}+\frac{\rho h}{\theta}. \tag{4.1}$$

Differentiating the classical thermodynamical relation

$$\psi = \epsilon-\eta\theta \tag{4.2}$$

with respect to time, we obtain the second expression

$$\rho\theta^{-1}\dot{\epsilon} = \rho\theta^{-1}\dot{\psi}+\rho\dot{\eta}+\rho\theta^{-1}\eta\dot{\theta}. \tag{4.3}$$

Eliminating $\dot{\epsilon}$ between equations (4.1) and (4.3) and introducing the recoverable and dissipative parts of the constitutive variables $\boldsymbol{t}$, $\boldsymbol{\mathscr{B}}$ and $\boldsymbol{\tau}$ by the relations

$$\boldsymbol{t} = {}^{\mathrm{R}}\boldsymbol{t}+{}^{\mathrm{D}}\boldsymbol{t}, \qquad \boldsymbol{\mathscr{B}} = {}^{\mathrm{R}}\boldsymbol{\mathscr{B}}+{}^{\mathrm{D}}\boldsymbol{\mathscr{B}}, \qquad \boldsymbol{\tau} = {}^{\mathrm{R}}\boldsymbol{\tau}+{}^{\mathrm{D}}\boldsymbol{\tau} \tag{4.4}$$

we obtain the equation of entropy production in the form

$$\rho\dot{\eta} = \mathscr{J}+\mathscr{C} \tag{4.5}$$

where the Jouguet and Clausius terms have been defined as (cf Germain 1967, Maugin 1971a)

$$\mathscr{J} = \rho\,{}^{\mathrm{R}}\dot{\eta}+\rho\phi$$
$$\mathscr{C} = (q_k/\theta)_{,k}+\frac{\rho h}{\theta} \tag{4.6}$$

in which the reversible entropy production $\rho\,{}^{\mathrm{R}}\dot{\eta}$ and the dissipation function per unit volume $\rho\phi$ are given by

$$\rho\,{}^{\mathrm{R}}\dot{\eta} \equiv \theta^{-1}({}^{\mathrm{R}}t_{kl}v_{k,l}+{}^{\mathrm{R}}\tau_{kl}\dot{\mu}_{k,l}-\rho\,{}^{\mathrm{R}}\mathscr{B}_k\dot{\mu}_k-\rho\eta\dot{\theta}-\rho\dot{\psi}) \tag{4.7}$$

$$\rho\phi \equiv \theta^{-1}({}^{\mathrm{D}}t_{kl}v_{k,l}+{}^{\mathrm{D}}\tau_{kl}\dot{\mu}_{k,l}-\rho\,{}^{\mathrm{D}}\mathscr{B}_k\dot{\mu}_k-\theta^{-1}q_k\theta_{,k}). \tag{4.8}$$

The first of these is identically equal to zero. Indeed we have shown in a preceding article (Maugin and Eringen 1972a) that the recoverable parts of the constitutive equations were derivable from a potential, the free energy density ψ. If the reversible behaviour of the solid under consideration is of the nonlinear elastic (hyperelastic) type, taking

$$\psi = \psi(x_{k,K}; \mu_k; \mu_{k,K}; \theta; \theta_{,K}) \tag{4.9}$$

and assuming independent dynamical processes (ie rates d_{kl}, π_{kl}, $\dot{\mu}_k$, $\dot{\mu}_{k,l}$, $\dot{\theta}$ and $\dot{\theta}_{,k}$), we obtained

$$^{\mathrm{R}}t_{(kl)} = \rho \frac{\partial \psi}{\partial x_{(k,K}} x_{l),K}, \qquad {}^{\mathrm{R}}\mathscr{B}_k = -\frac{\partial \psi}{\partial \mu_k} \tag{4.10}$$

$$^{\mathrm{R}}t_{[kl]} = \rho \frac{\partial \psi}{\partial x_{[k,K}} x_{l],K} = \rho\, {}^{\mathrm{R}}\mathscr{B}_{[k}\mu_{l]} \tag{4.11}$$

$$^{\mathrm{R}}\tau_{kl} = \rho \frac{\partial \psi}{\partial \mu_{k,K}} x_{l,K} \tag{4.12}$$

$$\eta = -\frac{\partial \psi}{\partial \theta}, \qquad \frac{\partial \psi}{\partial \theta_{,K}} = 0. \tag{4.13}$$

Thus,

$$\rho\, {}^{\mathrm{R}}\dot{\eta} = 0, \qquad \mathscr{J} \equiv \rho\phi. \tag{4.14}$$

Finally, on account of equations (4.5) and (4.14), the entropy inequality (3.10) yields

$$\rho\phi \geqslant 0 \tag{4.15}$$

with ϕ given by equation (4.8). Upon using the decomposition of $v_{k,l}$ in its symmetric and skewsymmetric parts, equations (3.2) and the result (3.6), we can write the dissipation $\rho\phi$ in the equivalent form

$$\rho\theta\phi = {}^{\mathrm{D}}t_{(kl)}d_{kl} + {}^{\mathrm{D}}\tau_{kl}\dot{\mu}_{k,l} - \rho\, {}^{\mathrm{D}}\mathscr{B}_k(\dot{\mu}_k - \pi_{lk}\mu_l) - \theta^{-1}q_k\theta_{,k} \tag{4.16}$$

which, in turn, can be concisely written as

$$\rho\theta\phi = \chi_{(\beta)}\dot{\alpha}_{(\beta)} \tag{4.17}$$

where $\dot{\alpha}_{(\beta)}, (\beta) = 1, \ldots, 21$ is the indexed series of independent components of the generalized fluxes whose set is formed by the quantities d_{kl}, $\dot{\mu}_{k,l}$, $\theta_{,k}$ and M_k, the latter being defined as

$$M_k = \dot{\mu}_k - \pi_{lk}\mu_l. \tag{4.18}$$

$\chi_{(\beta)}, (\beta) = 1, \ldots, 21$ is the indexed series of independent components of the generalized irreversible forces whose set is formed by the quantities ${}^{\mathrm{D}}t_{(kl)}$, ${}^{\mathrm{D}}\tau_{kl}$, $-\rho^{\mathrm{D}}\mathscr{B}_k$ and $-q_k/\theta$.

One can verify that the fields which constitute the set $\dot{\alpha}_{(\beta)}$ are objective, that is, their forms are invariant under frame transformations of the type (cf Eringen 1967, chap 5)

$$x^* = Q(t)x + b(t), \qquad t^* = t - a$$

$$QQ^{\mathrm{T}} = Q^{\mathrm{T}}Q = I, \qquad \det Q = +1 \tag{4.19}$$

in which $Q(t)$ is a proper orthogonal transformation in $\mathbb{E}^3$, $b(t)$ is a spatial translation and a represents a constant shift of time.

Now we must determine the generalized irreversible forces. According to the axiom of equipresence (cf Eringen 1967, chap 7), they must depend on the same set of independent variables as the free energy ψ (equation (4.9)) does, unless this is not allowed by some general principle. Moreover, they should depend on the generalized fluxes.

Therefore, we consider the behavioural laws

$$\begin{aligned}
{}^{\mathrm{D}}t_{(kl)} &= {}^{\mathrm{D}}t_{(kl)}(x_{k,K}; \mu_k; \mu_{k,K}; \theta|\theta_{,k}; d_{kl}; \dot{\mu}_{k,l}; M_k) \\
{}^{\mathrm{D}}\mathscr{B}_k &= {}^{\mathrm{D}}\mathscr{B}_k(\cdot\,;\cdot\,;\cdot\,;\cdot\,|\cdot\,;\cdot\,;\cdot\,;\cdot) \\
{}^{\mathrm{D}}\tau_{kl} &= {}^{\mathrm{D}}\tau_{kl}(\cdot\,;\cdot\,;\cdot\,;\cdot\,|\cdot\,;\cdot\,;\cdot\,;\cdot) \\
q_k &= q_k(\cdot\,;\cdot\,;\cdot\,;\cdot\,|\cdot\,;\cdot\,;\cdot\,;\cdot)
\end{aligned} \tag{4.20}$$

which we assume to be at least of class C^1 with respect to their last four arguments. Then, the inequality (4.15) clearly implies that

$$ {}^{\mathrm{D}}t_{(kl)}(\ldots|\theta_{,k}; \mathbf{0}; \dot{\mu}_{k,l}; M_k) = 0 \tag{4.21}$$

$$ {}^{\mathrm{D}}\mathscr{B}_k(\ldots|\theta_{,k}; d_{kl}; \dot{\mu}_{kl}; \mathbf{0}) = 0 \tag{4.22}$$

$$ {}^{\mathrm{D}}\tau_{kl}(\ldots|\theta_{,k}; d_{kl}; \mathbf{0}; M_k) = 0 \tag{4.23}$$

$$ q_k(\ldots|\mathbf{0}; d_{kl}; \dot{\mu}_{k,l}; M_k) = 0. \tag{4.24}$$

That is, a generalized irreversible force vanishes whenever the corresponding generalized flux vanishes.

Of importance for the following, is the principle of least irreversible force due to Ziegler (1963). In the n dimensional differentiable manifold $\mathscr{V}^n$ of the thermodynamical variables $\alpha_{(\beta)}$, $(\beta) = 1, \ldots, n$, the dissipation function

$$\rho\theta\phi = \Phi(\dot{\alpha}_{(\beta)}) \tag{4.25}$$

represents, for each prescribed value of the rate $\dot{\alpha}_{(\beta)}$, a hypersurface

$$\Phi(\dot{\alpha}_{(\beta)}) = M. \tag{4.26}$$

Assuming that the process considered is quasistatic, that is, the changes of the generalized coordinates $\alpha_{(\beta)}$ and the temperature are sufficiently slow, the principle of least irreversible force states that:

if the value $M > 0$ of the dissipation function $\Phi(\dot{\alpha}_{(\beta)})$ and the direction $\nu_{(\beta)}$ of the irreversible force ($\chi_{(\beta)} = \chi\nu_{(\beta)}$) are prescribed then, the actual quasistatic velocity $\dot{\alpha}_{(\beta)}$ minimizes the magnitude χ of the irreversible force $\chi_{(\beta)}$ subject to the condition $\Phi(\dot{\alpha}_{(\beta)}) \geqslant 0$.

A consequence of this principle is that the components of the irreversible forces are given by the relations (Ziegler 1963)

$$\begin{aligned}
\chi_{(\beta)} &= \lambda \frac{\partial \Phi}{\partial \dot{\alpha}_{(\beta)}} \\
\lambda &= \Phi\left(\frac{\partial \Phi}{\partial \dot{\alpha}_{(\beta)}} \dot{\alpha}_{(\beta)}\right)^{-1}.
\end{aligned} \tag{4.27}$$

If Φ is homogeneous of degree N in the variables $\dot{\alpha}_{(\beta)}$ then Euler's theorem yields

$$\lambda = N^{-1}. \tag{4.28}$$

The problem of finding the irreversible forces is then reduced to that of constructing an *ad hoc* function Φ. Therefore Φ plays, for the dissipative parts of the constitutive equations, a role equivalent to that played by the free energy ψ for the recoverable parts.

The study of irreversible processes could be carried along in the generality provided by the relations (4.20) however, we shall content ourselves with simple cases of special interest.

4.2. Magnetic spin–crystal lattice interactions

Consider

$$\Phi = \Phi_1(x_{k,K};\mu_k;\mu_{k,K};\theta|d_{kl};\theta_{,k})+\Phi_2(\dots|M_k;\theta_{,k})+\Phi_3(\dots|\dot{\mu}_{k,l};\theta_{,k}) \tag{4.29}$$

with, in agreement with the relations (4.21) through (4.24),

$$\Phi_1(\dots|\mathbf{0};\theta_{,k}) = 0 \tag{4.30}$$

$$\Phi_2(\dots|\mathbf{0};\theta_{,k}) = 0 \tag{4.31}$$

$$\Phi_3(\dots|\mathbf{0};\theta_{,k}) = 0. \tag{4.32}$$

As a special case, we consider that for which equations (4.30) and (4.32) hold true. Thus (4.21) and (4.23) hold true. Moreover we take†

$$\Phi = \Phi(\mu_S(\theta);\theta|M_k;\theta_{,k}), \qquad \mu_S = |\boldsymbol{\mu}|. \tag{4.33}$$

This means that irreversibility is mainly due to heat conduction and interactions between electron spins and the crystal lattice (see the physical significance granted to the field $\mathscr{B}$ in Maugin and Eringen (1972a), also Tiersten (1964)). The inequality (4.15) is now reduced to

$$-\rho\, {}^{\mathrm{D}}\mathscr{B}_k M_k - \theta^{-1} q_k \theta_{,k} \geqslant 0. \tag{4.34}$$

Spatial isotropy requires the scalar function be dependent on its vectorial arguments only through their inner product (cf Eringen 1967, Appendix B.6)

$$I_1 = M_k M_k, \qquad I_2 = M_k \theta_{,k}, \qquad I_3 = \theta_{,k}\theta_{,k}. \tag{4.35}$$

But, since M_k is axial and the required invariance is that under the full orthogonal group, we must have

$$\frac{\partial \Phi}{\partial I_2} = 0. \tag{4.36}$$

We remark that, dealing with magnetic processes, we should also consider the time symmetry $\mathscr{R}$ (cf Maugin and Eringen 1972b, Maugin 1972a) as a required invariance. The final form of Φ which follows from equations (4.33), (4.35) and (4.36),

$$\Phi = \Phi(\mu_S(\theta);\theta;I_1;I_3), \tag{4.37}$$

a homogeneous function of degree N_1 and N_3 in M_k and $\theta_{,k}$ respectively, satisfies this invariance since $\boldsymbol{\mu}$ changes sign with the current and hence with time. After (4.27), we have

$$\begin{aligned} -\rho\, {}^{\mathrm{D}}\mathscr{B}_k &= \frac{2}{N_1}\frac{\partial \Phi}{\partial I_1} M_k \\ -\theta^{-1} q_k &= \frac{2}{N_3}\frac{\partial \Phi}{\partial I_3}\theta_{,k}. \end{aligned} \tag{4.38}$$

† The relation $\mu_s = \mu_s(\theta)$ resorts to microscopic physics.

A simple approximation to (4.37) is

$$\Phi = A(\mu_S, \theta)I_1 + B(\mu_S, \theta)I_3 \tag{4.39}$$

with

$$\begin{aligned} N_1 &= N_3 = 2 \\ A &= \alpha(\mu_S, \theta)\mu_S^{-1}\rho_R, \qquad \alpha > 0 \\ B &= \kappa(\mu_S, \theta)\theta^{-1}, \qquad \kappa > 0. \end{aligned} \tag{4.40}$$

Then,

$$^{D}\mathscr{B}_k = -\alpha(\mu_S, \theta)\mu_S^{-1}JM_k \tag{4.41}$$

$$q_k = -\kappa(\mu_S, \theta)\theta_{,k} \tag{4.42}$$

with

$$J = \det(x_{k,K}) = \rho_R/\rho \tag{4.43}$$

which is the solution to the continuity equation (3.4). The inequality (4.34) is obviously verified. Equation (4.42) is Fourier's equation of conduction with κ the conduction coefficient. α is nothing but the damping coefficient referred to in § 2. Indeed carrying the second of the expressions (4.4) in equation (3.7) and taking account of the results (4.41) and the definition (4.18), we obtain (T = transposed)

$$\dot{\boldsymbol{\mu}} = \gamma_0(\boldsymbol{B} + {}^{R}\boldsymbol{\mathscr{B}} + \rho^{-1}\nabla \,.\, {}^{R}\boldsymbol{\tau}) \times \boldsymbol{\mu} - \alpha(\mu_S, \theta)\mu_S^{-1}\gamma_0 J\{\dot{\boldsymbol{\mu}} \times \boldsymbol{\mu} - (\boldsymbol{\pi}^T \,.\, \boldsymbol{\mu}) \times \boldsymbol{\mu}\}. \tag{4.44}$$

This is Gilbert and Kelley's equation generalized to deformable media since, if there were no deformation

$$J = 1, \qquad \boldsymbol{\pi} = 0 \tag{4.45}$$

equation (4.44) would reduce to

$$\dot{\boldsymbol{\mu}} = \gamma_0\, {}^{R}\boldsymbol{B}^{\text{eff}} \times \boldsymbol{\mu} - \alpha(\mu_S, \theta)\gamma_0\mu_S^{-1}\dot{\boldsymbol{\mu}} \times \boldsymbol{\mu}. \tag{4.46}$$

Landau and Lifshitz's equation would then be obtained by the approximation mentioned in § 2. We remark that the dissipation function,

$$\Phi' = \alpha\mu_S^{-1}\rho_R M_k M_k, \tag{4.47}$$

is very similar to the Rayleigh dissipation function considered by Gilbert and Kelley (1955, see also Brown 1963).

4.2.1. Remark. Clearly, the principle of least irreversible force used above is a generalization of Onsager's relations which, in contrast to the present principle deal exclusively with linear processes. A direct application of Onsager's relations would, of course, yield results identical to those of equations (4.41) and (4.42). In fact, these relations require that

$$\begin{aligned} -\rho\, {}^{D}\mathscr{B}_k &= A_{kl}M_l \\ -\theta^{-1}q_k &= B_{kl}\theta_{,l} \end{aligned} \tag{4.48}$$

and isotropy gives, δ_{kl} being the unit tensor,

$$A_{kl} = A\delta_{kl}, \qquad B_{kl} = B\delta_{kl}. \tag{4.49}$$

Taking

$$A = \alpha\mu_S^{-1}\rho_R, \qquad B = \kappa\theta^{-1},$$

we obtain equations (4.41)–(4.42). The identity with Onsager's procedure started with the approximation (4.39). Speculations on more general forms of the expression (4.37) would certainly bring interesting new features.

4.3. *More complex dissipative phenomena*

Following the scheme proposed in § 4.1, we may consider more involved irreversible phenomena. In particular, if we do not take the function Φ_1 of equation (4.29) equal to zero, we may construct stress constitutive equations for a class of viscoelastic solids (eg, see the works of Gorodtsov and Leonov 1968 and Piau 1970). However we are mainly interested here in magnetic phenomena. Hence we keep Φ_1 equal to zero.

In agreement with the physical significance given to the tensor τ (cf Maugin and Eringen 1972a, Tiersten 1964 and Brown 1966), if spin–spin interactions are the predominant source of dissipation and relaxation of the spin, we can take

$$\Phi_2 = 0$$

and write

$$\Phi = \Phi(\,.\,;.\,;.\,;.\,|\dot{\mu}_{k,l};\theta_{,k}). \tag{4.50}$$

More specifically,

$$\Phi = \Phi(\mu_S(\theta), \theta, I_3, I_4) \tag{4.51}$$

which is a homogeneous function of degree N_3 and N_4 in $\theta_{,k}$ and $\dot{\mu}_{k,l}$ respectively. It is a scalar invariant under the group of transformations (4.19). We have defined

$$I_4 = \dot{\mu}_{k,l}\dot{\mu}_{k,l}. \tag{4.52}$$

A simple approximation to (4.51) is

$$\Phi = C(\mu_S)I_4 + B(\mu_S, \theta)I_3 \tag{4.53}$$

$$N_3 = N_4 = 2$$

$$B = \kappa(\mu_S, \theta)\theta^{-1}, \qquad \kappa > 0 \tag{4.54}$$

$$C = \delta(\mu_S)\mu_S^{-1}L^2\rho_R, \qquad \delta > 0$$

in which δ has the same dimension as the damping coefficient α of § 4.2. L is a characteristic length, for instance, the lattice constant or either a typical magnetic domain size or the thickness of a Bloch wall. According to a remark made in § 2, in the present case, relaxation phenomena are quite independent of temperature. Therefore δ is taken to be independent of θ. We then have

$$^{D}\tau_{kl} = \frac{2}{N_4}\frac{\partial\Phi}{\partial I_4}\dot{\mu}_{k,l}, \qquad q_k = -\frac{2\theta}{N_3}\frac{\partial\Phi}{\partial I_3}\theta_{,k} \tag{4.55}$$

hence

$$\begin{aligned} ^{D}\tau_{kl} &= \delta(\mu_S)\mu_S^{-1}L^2\rho_R\dot{\mu}_{k,l} \\ q_k &= -\kappa(\mu_S, \theta)\theta_{,k}. \end{aligned} \tag{4.56}$$

Carrying the first of these expressions in equation (3.7), we obtain the spin equation when spin–spin interactions provide the largest part of dissipation

$$\gamma_0^{-1}\dot{\boldsymbol{\mu}} = (\boldsymbol{B}+\mathscr{B}+\rho^{-1}\nabla\,.\,{}^{\mathrm{R}}\boldsymbol{\tau})\times\boldsymbol{\mu}+\delta(\mu_{\mathrm{S}})\mu_{\mathrm{S}}^{-1}JL^2(\nabla^2\dot{\boldsymbol{\mu}})\times\boldsymbol{\mu} \tag{4.57}$$

where ∇^2 is the Laplacian. For rigid solids, $J = 1$ (cf equation (4.45)).

Finally, if we consider dissipation which results from both spin–crystal lattice and spin–spin interactions, taking the following simple dissipation function quadratic in the generalized fluxes:

$$\Phi = A(\mu_{\mathrm{S}},\theta)I_1+B(\mu_{\mathrm{S}},\theta)I_3+C(\mu_{\mathrm{S}})I_4 \tag{4.58}$$

we will obtain the complicated spin equation in deformable media

$$\begin{aligned}\gamma_0^{-1}\dot{\boldsymbol{\mu}} &= (\boldsymbol{B}+{}^{\mathrm{R}}\mathscr{B}+\rho^{-1}\nabla\,.\,{}^{\mathrm{R}}\boldsymbol{\tau})\times\boldsymbol{\mu}-\alpha(\mu_{\mathrm{S}},\theta)\mu_{\mathrm{S}}^{-1}J\{\dot{\boldsymbol{\mu}}\times\boldsymbol{\mu}-(\boldsymbol{\pi}^{\mathrm{T}}\,.\,\boldsymbol{\mu})\times\boldsymbol{\mu}\}\\ &\quad+\delta(\mu_{\mathrm{S}})\mu_{\mathrm{S}}^{-1}JL^2(\nabla^2\dot{\boldsymbol{\mu}})\times\boldsymbol{\mu}\end{aligned} \tag{4.59}$$

while the heat conduction law will assume Fourier's form (cf equation (4.56, part two)). The equation of heat propagation which corresponds to the spin equation (4·59) is established by the usual method (cf Eringen 1967, p 179). Replace $\dot{\epsilon}$ in equation (3.9) by its value given by equation (4.3). Then, with ψ given by equation (4.9) subject to the restriction provided by the second of equations (4.13), compute $\dot{\eta}$ from the definition (4.13, part one) of η. One obtains

$$\dot{\eta} = -\overline{\left(\frac{\partial\psi}{\partial\theta}\right)}^{\boldsymbol{\cdot}} = -\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta}-\frac{\partial^2\psi}{\partial\theta\,\partial x_{k,K}}x_{l,K}v_{k,l}-\frac{\partial^2\psi}{\partial\theta\,\partial\mu_k}\dot{\mu}_k-\frac{\partial^2\psi}{\partial\theta\,\partial\mu_{k,K}}x_{l,K}\dot{\mu}_{k,l}. \tag{4.60}$$

The right hand side of equation (3.9) is transformed by the aid of equations (4.4) and the results (4.10)–(4.12) and (4.56). One readily gets the final result

$$\mathscr{D}\theta+P_{kl}d_{kl}+Q_k(\dot{\mu}_k-\pi_{lk}\mu_l)+R_{kl}\dot{\mu}_{k,l}+S = 0 \tag{4.61}$$

with

$$\begin{aligned}&S \equiv \Phi-B(\mu_{\mathrm{S}},\theta)I_3+\rho h\\ &P_{kl} \equiv \rho\theta\frac{\partial^2\psi}{\partial\theta\,\partial x_{(k,K}}x_{l),K}, \qquad Q_k \equiv \rho\theta\frac{\partial^2\psi}{\partial\theta\,\partial\mu_k}\\ &R_{kl} \equiv \rho\theta\frac{\partial^2\psi}{\partial\theta\,\partial\mu_{k,K}}x_{l,K}.\end{aligned} \tag{4.62}$$

$\mathscr{D}$ is the parabolic differential operator defined as

$$\mathscr{D} \equiv \rho C_h\left(\frac{\partial}{\partial t}+\boldsymbol{v}\,.\,\nabla\right)-\kappa(\mu_{\mathrm{S}},\theta)\nabla^2 \tag{4.63}$$

with

$$C_h \stackrel{\mathrm{def}}{=} \theta\frac{\partial^2\psi}{\partial\theta^2} > 0, \qquad \kappa > 0. \tag{4.64}$$

C_h is the specific heat at constant deformation, magnetization and magnetization gradient fields; $(\kappa/\rho C_h)$ is the thermal diffusibility. Equation (4.61) is the equation of heat propagation in a magnetically saturated nonlinear elastic solid with linear dissipation of micromagnetic origin (no mechanical dissipation, cf equation (4.58)).

5. Conclusion

It remains to determine experimentally or from a microscopical approach the value of the unknown coefficients α and δ and to show that equation (4.59) provides a good description of the approach of $\boldsymbol{\mu}$ into parallelism with the field $\boldsymbol{B}^{\text{eff}}$. This can only be achieved by comparing the theory given here with experimental results.

In a forthcoming publication (Maugin 1972c), we shall study dissipative phenomena in the relativistically invariant theory of micromagnetism developed by Maugin and Eringen (1972c) (also Maugin 1972b) by using relativistic thermodynamics (cf Maugin 1971a).

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